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A new integrable differential-difference system and its explicit solutions

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Abstract. A new integrable differential-difference system is proposed. By the dependent variable transformation, the system is transformed into multilinear form. By introducing an auxiliary variable, we further transform it into the bilinear form. Its corresponding Bäcklund transformation is obtained. Furthermore, nonlinear superposition formulae are presented. As an application of the obtained results, soliton solutions and rational solutions to the system are derived.

1. Introduction

There has been considerable interest in searching for new integrable discrete systems and studying their properties. Many papers have been dedicated to the subject, and various approaches are currently available. Two of them are Hirota's method and the Bäcklund transformation [1–8]. In this paper, we will propose a new differential-difference system and then study it by using the Hirota method. By the dependent variable transformation, the system under consideration is transformed into multilinear equations. By introducing an auxiliary variable, we further decouple it into the bilinear form. The corresponding Bäcklund transformation is found and nonlinear superposition formulae are established. As a result, multisoliton solutions and rational solutions are derived.

It has been noted recently that the so-called Belov–Chaltikian lattice [9]

$$b_t(n) = b(n)\,(b(n+1) - b(n-1)) - c(n) + c(n-1) \tag{1}$$

$$c_t(n) = c(n)(b(n+2) - b(n-1)) \tag{2}$$

and the Blaszak–Marciniak lattice [10]

$$a_t(n) = c(n+1) - c(n-1) \tag{3}$$

$$b_t(n) = a(n-1)\,c(n-1) - a(n)\,c(n) \tag{4}$$

$$c_t(n) = c(n)(b(n) - b(n+1)) \tag{5}$$

are transformed into the following bilinear forms [11, 12]:

$$(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}) f(n) \cdot f(n) = 0 \tag{6}$$

$$(D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}) f(n) \cdot f(n) = 0 \tag{7}$$

and

$$(D_t^2 - 2D_z e^{D_n}) f(n) \cdot f(n) = 0 \tag{8}$$

$$(D_z D_t - 4\sinh^2(\tfrac{1}{2}D_n)) f(n) \cdot f(n) = 0 \tag{9}$$

respectively, where z is an auxiliary variable and the bilinear operators are defined as follows [2–6]:

$$D_z^m D_t^k\, a \cdot b \equiv \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(z,t)\, b(z',t')\bigg|_{z'=z,t'=t}$$

$$\exp(\delta D_n)\, a(n) \cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)\, b(n')\bigg|_{n'=n} = a(n+\delta)\, b(n-\delta).$$

On the other hand, in [13], a new and relatively simple procedure for finding new integrable differential-difference equations was reported. By combining these two concepts, it is natural to search for new integrable systems such that the systems have bilinear forms of the type (6), (7) and (8), (9) and the corresponding bilinear Bäcklund transformations could be found. With such a motivation in mind, and after some tests and guesswork, we now propose the following new system:

$$a_t(n) = \frac{c(n+1)}{c(n)} - \frac{c(n)}{c(n-1)} \tag{10}$$

$$b_t(n+1) + b_t(n) = a(n) - (b(n+1) - b(n))^2 \tag{11}$$

$$c_t(n) = c(n)(b(n+1) - b(n)). \tag{12}$$

Set $b(n) = (\ln f(n))_t$. Then from (12) we obtain

$$(\ln c(n))_t = \left(\ln \frac{f(n+1)}{f(n)}\right)_t$$

i.e.

$$c(n) = c_0(n) \frac{f(n+1)}{f(n)}$$

where $c_0(n)$ is some function of n. In the following, we just consider the case $c_0(n) =$ constant, which corresponds to the soliton case. In this case, we may choose $c_0 = 1$ without loss of generality. Besides, from (11) we have

$$a(n) = \frac{D_t^2 f(n+1) \cdot f(n)}{f(n+1)\, f(n)}.$$

Thus, by the dependent variable transformation

$$a(n) = \frac{D_t^2 f(n+1) \cdot f(n)}{f(n+1)\, f(n)} \qquad b(n) = (\ln f(n))_t \qquad c(n) = \frac{f(n+1)}{f(n)} \tag{13}$$

equation (10) can be transformed into the following form:

$$\left[\frac{D_t^2 f(n+1) \cdot f(n)}{f(n+1)\, f(n)}\right]_t = \frac{f(n+2) f(n)}{f^2(n+1)} - \frac{f(n+1) f(n-1)}{f^2(n)} \tag{14}$$

or, equivalently,

$$D_t(D_t^2 e^{\frac{1}{2}D_n} f(n) \cdot f(n)) \cdot (e^{\frac{1}{2}D_n} f(n) \cdot f(n)) = 2\sinh(\tfrac{1}{2}D_n)[e^{D_n} f(n) \cdot f(n)] \cdot f^2(n). \tag{15}$$

By introducing an auxiliary variable z and using (A1), we can decouple (15) into the following bilinear form:

$$(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n}) f(n) \cdot f(n) = 0 \tag{16}$$

$$(D_z D_t - 4\sinh^2(\tfrac{1}{2}D_n)) f(n) \cdot f(n) = 0. \tag{17}$$

Reversing the process, we know that if $f(n)$ is a solution of (16) and (17), then $a(n)$, $b(n)$ and $c(n)$, given by (13), satisfy (10)–(12). It is remarked that, in the continuous case, the technique to decouple multilinear equations into bilinear ones by introducing auxiliary variables was first proposed by Hirota and Satsuma. They applied it to the Lax fifth-order KdV equation and a model equation for shallow water waves [2, 14].

The system (10)–(12) can be rewritten as the following equation for a single field:

$$\begin{aligned} &u_{ttt}(n+1) + u_{ttt}(n) + 2(u_t(n+1) - u_t(n))(u_{tt}(n+1) - u_{tt}(n)) \\ &\qquad = e^{u(n+2)-2u(n+1)+u(n)} - e^{u(n+1)-2u(n)+u(n-1)} \end{aligned} \tag{18}$$

where $b(n) = u_t(n)$. It is noted that the highest derivative with respect to t appearing in (18) contains two terms taking values at n and $n+1$, respectively. Therefore, equation (18) may be viewed as a nonlocal differential-difference equation in this sense. However, to our knowledge, most integrable differential-difference systems appearing in the literature such as the Toda lattice or the Volterra equation are local. Therefore, it is far from obvious how to find such a transformation if there exists some transformation (point transformation, Miura-like transformation, etc) which relates equation (18) or the system (10)–(12) to some other one which has already appeared in the literature. On the other hand, from the viewpoint of bilinear formalism, to our knowledge it is the first time that the bilinear equations (16) and (17) have been considered simultaneously with z being an auxiliary variable, although (17) is just the bilinear form of the two-dimensional Toda lattice and can be obtained from the Hirota–Miwa equation $[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)] f \cdot f = 0$ (Z_i are arbitrary constants, $f = f(x_1, x_2, x_3)$, $D_i = D_{x_i}$) by reduction [15–17]. Based on these explanations, it would be reasonable to view (10)–(12) or (18) as a new system or equation.

The paper is organized as follows. In section 2 we give a Bäcklund transformation for equations (16) and (17) and then a nonlinear superposition formula. Soliton solutions of equations (16) and (17) are then found using this formula. In section 3, another nonlinear superposition formula is established. As an application of the obtained result, a sequence of polynomial solutions of (16) and (17) or rational solutions of (10)–(12) are derived. Finally, conclusions and discussions are given in section 4. The appendix lists some bilinear operator identities used in this paper.

2. A Bäcklund transformation, nonlinear superposition formula and soliton solutions

In this section, we shall give a Bäcklund transformation and nonlinear superposition formula for (16) and (17). First, by application of the exchange formalism, one can construct the following Bäcklund transformation for equations (16) and (17):

$$(D_t + \lambda^{-1} e^{-D_n} + \mu)\, f(n) \cdot g(n) = 0 \tag{19}$$

$$(D_z e^{-\frac{1}{2}D_n} - \lambda e^{\frac{1}{2}D_n} + \gamma e^{-\frac{1}{2}D_n})\, f(n) \cdot g(n) = 0 \tag{20}$$

$$(D_z - \lambda^{-1} D_t e^{-D_n} - \lambda^{-1}\mu e^{-D_n} - \omega)\, f(n) \cdot g(n) = 0 \tag{21}$$

where λ, μ, γ and ω are arbitrary constants. Furthermore, we can show the following result.

Proposition. *Let f_0 be a solution of equations (16) and (17) and suppose that f_i ($i = 1, 2$) are solutions of (16) and (17) which are related to f_0 under the BT equations (19)–(21) with* *parameters* $(\lambda_i, \mu_i, \gamma_i, \omega_i)$, *i.e.* $f_0 \overset{(\lambda_i,\mu_i,\gamma_i,\omega_i)}{\longrightarrow} f_i\ (i = 1, 2),\ \lambda_1\lambda_2 \neq 0,\ f_j \neq 0\ (j = 0, 1, 2)$. *Then* f_{12}, *defined by*

$$\exp\left(-\tfrac{1}{2}D_n\right) f_0 \cdot f_{12} = c\left[\lambda_1 \exp\left(-\tfrac{1}{2}D_n\right) - \lambda_2 \exp\left(\tfrac{1}{2}D_n\right)\right] f_1 \cdot f_2 \qquad (c \text{ is a nonzero constant}) \tag{22}$$

is a new solution which is related to f_1 *and* f_2 *under the BT (19)–(21) with parameters* $(\lambda_2, \mu_2, \gamma_2, \omega_2)$ *and* $(\lambda_1, \mu_1, \gamma_1, \omega_1)$, *respectively.*

As an application of the result, we can construct soliton solutions of (16) and (17). Choose, for example, $f_0 = 1$ and $c = 1/(\lambda_1 - \lambda_2)$. It is easily verified that

$$\begin{tikzcd}
 & 1 + e^{\eta_1} \arrow[dr, "(\lambda_2,\mu_2,\gamma_2,\omega_2)"] & \\
1 \arrow[ur, "(\lambda_1,\mu_1,\gamma_1,\omega_1)"] \arrow[dr, "(\lambda_2,\mu_2,\gamma_2,\omega_2)"'] & & f_{12} \\
 & 1 + e^{\eta_2} \arrow[ur, "(\lambda_1,\mu_1,\gamma_1,\omega_1)"'] &
\end{tikzcd}$$

where

$$f_{12} = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_2}{\lambda_1 - \lambda_2} e^{\eta_1} + \frac{\lambda_1 - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_2} + \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_1 + \eta_2}$$

with

$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0 \qquad q_i = \lambda_i(1 - e^{-p_i}) \qquad r_i = \lambda_i^{-1}(e^{p_i} - 1)$$

and

$$\lambda_i = [e^{p_i}(1 + e^{p_i})]^{1/3} \qquad \mu_i = -\lambda_i^{-1} \qquad \gamma_i = \lambda_i \qquad \omega_i = \lambda_i^{-2}.$$

In general, along this line, we can obtain multisoliton solutions for (16) and (17) step by step. By the dependent variable transformation (13), we can obtain the corresponding soliton solutions of (10)–(12). For example,

$$a(n) = \frac{(e^p + 1)r^2 e^\eta}{(1 + e^\eta)(1 + e^p e^\eta)} \qquad b(n) = \frac{r e^\eta}{1 + e^\eta} \qquad c(n) = \frac{1 + e^p e^\eta}{1 + e^\eta}$$

is the one-soliton solution of (10)–(12), where $\eta = pn + qz + rt + \eta^0$, $q = \lambda(1 - e^{-p})$, $r = \lambda^{-1}(e^p - 1)$, $\lambda = [e^p(1 + e^p)]^{1/3}$ with η^0 being an arbitrary constant. Since equations (10)–(12) do not involve the extra variable z, we may view z as an arbitrary parameter and qz appearing in η can be absorbed by η^0.

Set $\psi_n = f(n)/g(n)$, $u(n) = \ln g(n)$. Then, from the bilinear BT (19)–(21) and by some calculations, we can obtain the following Lax pair for (18):

$$\begin{aligned}
\psi_{n+1} + \lambda^{-1}[u_{tt}(n+1) + u_{tt}(n) + (u_t(n+1) - u_t(n))^2 - \gamma - \omega]\psi_n \\
= \lambda^{-2}(u_t(n-1) - u_t(n+1))\, e^{u(n+1)-2u(n)+u(n-1)}\psi_{n-1} \\
-\lambda^{-3} e^{u(n+1)-u(n)-u(n-1)+u(n-2)}\psi_{n-2}
\end{aligned} \tag{23}$$

$$\psi_{n_t} + \lambda^{-1} e^{u(n+1)-2u(n)+u(n-1)}\psi_{n-1} + \mu\psi_n = 0. \tag{24}$$

3. Nonlinear superposition formula and rational solutions

We now turn to consider rational solutions of (10)–(12) or polynomial solutions of (16) and (17). In order to obtain polynomial solutions of (16) and (17), it is enough to consider the special Bäcklund parameters of (19)–(21), i.e. $\lambda = 2^{1/3}$, $\mu = -2^{-1/3}$, $\gamma = 2^{1/3}$ and $\omega = 2^{-2/3}$. In this case, equations (19)–(21) become

$$(D_t + 2^{-1/3}\mathrm{e}^{-D_n} - 2^{-1/3}) f(n) \cdot g(n) = 0 \tag{25}$$

$$(D_z \mathrm{e}^{-D_n/2} - 2^{1/3}\mathrm{e}^{D_n/2} + 2^{1/3}\mathrm{e}^{-D_n/2}) f(n) \cdot g(n) = 0 \tag{26}$$

$$(D_z - 2^{-1/3} D_t \mathrm{e}^{-D_n} + 2^{-2/3}\mathrm{e}^{-D_n} - 2^{-2/3}) f(n) \cdot g(n) = 0. \tag{27}$$

We shall represent the transformation (25)–(27) symbolically by $f(n) \longrightarrow g(n)$. Now let $f_0(n)$, $f_1(n)$ and $f_{12}(n)$ be three solutions of (16) and (17) and $f_0(n) \longrightarrow f_1(n) \longrightarrow f_{12}(n)$, with $f_0(n), f_1(n), f_{12}(n) \neq 0$. Suppose that $f_2(n)$ is given by

$$\exp\left(-\tfrac{1}{2}D_n\right) f_0 \cdot f_{12} = c \sinh\left(\tfrac{1}{2}D_n\right) f_1 \cdot f_2 \qquad (c \text{ is a nonzero constant}). \tag{28}$$

From these assumptions and by use of (A2)–(A6) and (28), we have

$$\sinh(\tfrac{1}{2}D_n)[c D_t f_1(n) \cdot f_2(n) - 2^{2/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n)] \cdot f_1^2(n) = 0 \tag{29}$$

$$\sinh(\tfrac{1}{2}D_n)[c D_z f_1(n) \cdot f_2(n) - 2^{4/3} f_0(n) f_{12}(n)] \cdot f_1^2(n) = 0 \tag{30}$$

$$\begin{aligned}\sinh(\tfrac{1}{2}D_n)[c D_z f_1(n) \cdot f_2(n) &+ 2^{2/3} D_t \mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) \\ &-2^{4/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n)] \cdot f_1^2(n) = 0\end{aligned} \tag{31}$$

i.e.

$$c D_t f_1(n) \cdot f_2(n) - 2^{2/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) = c_1(t,z) f_1^2(n) \tag{32}$$

$$c D_z f_1(n) \cdot f_2(n) - 2^{4/3} f_0(n) f_{12}(n) = c_2(t,z) f_1^2(n) \tag{33}$$

$$c D_z f_1(n) \cdot f_2(n) + 2^{2/3} D_t \mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) - 2^{4/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) = c_3(t,z) f_1^2(n) \tag{34}$$

where $c_i(t,z)$ $(i = 1, 2, 3)$ are suitable functions of t and z. Furthermore, we assume that $f_2(n)$ determined by (28) is chosen such that $c_i(t,z) = 0$, $i = 1, 2, 3$. In this case, we have

$$c D_t f_1(n) \cdot f_2(n) - 2^{2/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) = 0 \tag{35}$$

$$c D_z f_1(n) \cdot f_2(n) - 2^{4/3} f_0(n) f_{12}(n) = 0 \tag{36}$$

$$c D_z f_1(n) \cdot f_2(n) + 2^{2/3} D_t \mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) - 2^{4/3}\mathrm{e}^{-D_n} f_0(n) \cdot f_{12}(n) = 0. \tag{37}$$

By use of (28), (35)–(37) and (A7)–(A9), we can deduce that

$$(D_t + 2^{-1/3}\mathrm{e}^{-D_n} - 2^{-1/3}) f_0(n) \cdot f_2(n) = 0 \tag{38}$$

$$(D_z \mathrm{e}^{-\frac{1}{2}D_n} - 2^{1/3}\mathrm{e}^{\frac{1}{2}D_n} + 2^{1/3}\mathrm{e}^{-\frac{1}{2}D_n}) f_0(n) \cdot f_2(n) = 0 \tag{39}$$

$$(D_z - 2^{-1/3} D_t \mathrm{e}^{-D_n} + 2^{-2/3}\mathrm{e}^{-D_n} - 2^{-2/3}) f_0(n) \cdot f_2(n) = 0. \tag{40}$$

Therefore, $f_2(n)$ is a new solution and $f_0(n) \longrightarrow f_2(n)$. Similarly, we can show that $f_2(n) \longrightarrow f_{12}(n)$.

To summarize, we can seek particular solutions of (16) and (17) via the following steps. First, choose a given solution $f_1(n)$ of (16) and (17). Secondly, from the Bäcklund transformation (25)–(27) we find $f_0(n)$ and $f_{12}(n)$ such that $f_0(n) \longrightarrow f_1(n) \longrightarrow f_{12}(n)$ and, furthermore, obtain a particular solution $\tilde{f_2}(n)$ from (28). Then a general solution of (28) is $f_2(n) = \tilde{f_2}(n) + k(t,z) f_1(n)$, where $k(t,z)$ is an arbitrary function of t, z. Finally, we substitute $f_2(n)$ into (32)–(34). If $k(t,z)$ can be determined such that $c_i(t,z) = 0$, $i = 1, 2, 3$, the corresponding $f_2(n)$ is a new solution of (16) and (17). As an application of this result, we can obtain a sequence of polynomial solutions of (16) and (17). For example, if we choose $f_0(n) = n + 2^{1/3}z + 2^{-1/3}t + \alpha + \frac{4}{3}$, $f_1(n) = 1$ and $f_{12}(n) = n + 2^{1/3}z + 2^{-1/3}t + \alpha$, where α is a constant, then it is easily verified that $n + 2^{1/3}z + 2^{-1/3}t + \alpha + \frac{4}{3}$, $n + 2^{1/3}z + 2^{-1/3}t + \alpha$ and 1 are solutions of (16) and (17) and $n + 2^{1/3}z + 2^{-1/3}t + \alpha + \frac{4}{3} \longrightarrow 1 \longrightarrow n + 2^{1/3}z + 2^{-1/3}t + \alpha$. Then we seek a solution in the form

$$f_2(n) = (n + 2^{1/3}z + 2^{-1/3}t)^3 + A_1(t,z)(n + 2^{1/3}z + 2^{-1/3}t)^2 \\ +A_2(t,z)(n + 2^{1/3}z + 2^{-1/3}t) + A_3(t,z)$$

such that (28), (35), (36) and (37) hold. A direct calculation shows that

$$c = -\tfrac{2}{3} \qquad A_1 = 3\alpha + 2 \qquad A_2 = 3\alpha^2 + 4\alpha + 1 \qquad A_3 = -2^{1/3}z + c_0$$

where c_0 is an arbitrary constant. In this way, we may deduce a sequence of polynomial solutions of (16) and (17) and thus the rational solutions of (10)–(12).

4. Conclusion and discussion

In this paper, a new integrable differential-difference system is proposed. By the dependent variable transformation, the system is transformed into multilinear form. By introducing an auxiliary variable, we further transform it into bilinear form. A corresponding Bäcklund transformation for it is obtained. Furthermore, nonlinear superposition formulae are presented. As an application of the obtained results, soliton solutions and rational solutions to the system are derived. Besides, it would be of interest to study the integrable continuous limit and integrable full discretization for the system (10)–(12) or equation (18). Moreover, we can further consider the following extended form of (16) and (17):

$$(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n} + \tfrac{1}{2}\alpha^2(e^{\frac{3}{2}D_n} - e^{\frac{1}{2}D_n}))\, f(n)\cdot f(n) = 0 \tag{41}$$

$$(D_z D_t + \alpha D_z \sinh(D_n) - 4\sinh^2(\tfrac{1}{2}D_n))\, f(n)\cdot f(n) = 0 \tag{42}$$

where α is a constant. The following BT for (41) and (42) is found:

$$\left(D_t + \lambda^{-1}e^{-D_n} - \tfrac{1}{4}\alpha^2\lambda e^{D_n} + \mu\right) f(n)\cdot g(n) = 0 \tag{43}$$

$$\left(D_z e^{-\frac{1}{2}D_n} + \tfrac{1}{2}\alpha\lambda D_z e^{\frac{1}{2}D_n} - \left(\lambda - \tfrac{1}{2}\alpha\lambda\gamma\right)e^{\frac{1}{2}D_n} + \gamma e^{-\frac{1}{2}D_n}\right) f(n)\cdot g(n) = 0 \tag{44}$$

$$\left(D_z - \lambda^{-1}D_t e^{-D_n} - \tfrac{1}{4}\alpha^2\lambda D_t e^{D_n} - \lambda^{-1}\mu e^{-D_n} - \tfrac{1}{4}\alpha^2\lambda\mu e^{D_n} - \omega\right) f(n)\cdot g(n) = 0 \tag{45}$$

where λ, μ, γ and ω are arbitrary constants.

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Appendix. Hirota bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$D_t[D_z e^{\frac{1}{2}D_n} a(n)\cdot a(n)]\cdot[e^{\frac{1}{2}D_n} a(n)\cdot a(n)] = \sinh(\tfrac{1}{2}D_n)[D_t D_z a(n)\cdot a(n)]\cdot a^2(n) \tag{A1}$$

$$\sinh(\tfrac{1}{2}D_n)(D_t a\cdot b)\cdot a^2 = D_t[\sinh(\tfrac{1}{2}D_n)a\cdot b]\cdot[e^{\frac{1}{2}D_n} a\cdot a] \tag{A2}$$

$$D_t[e^{-\frac{1}{2}D_n} a\cdot b]\cdot[e^{\frac{1}{2}D_n} c\cdot c] = e^{-\frac{1}{2}D_n}[(D_t a\cdot c)\cdot cb - ac\cdot(D_t c\cdot b)] \tag{A3}$$

$$2\sinh(\tfrac{1}{2}D_n)[e^{-D_n} a\cdot b]\cdot c^2 = e^{-\frac{1}{2}D_n}[ac\cdot(e^{-D_n} c\cdot b) - (e^{-D_n} a\cdot c)\cdot cb] \tag{A4}$$

$$2\sinh(\tfrac{1}{2}D_n)ab\cdot c^2 = (e^{\frac{1}{2}D_n} a\cdot c)(e^{-\frac{1}{2}D_n} c\cdot b) - (e^{-\frac{1}{2}D_n} a\cdot c)(e^{\frac{1}{2}D_n} c\cdot b) \tag{A5}$$

$$2\sinh(\tfrac{1}{2}D_n)[D_t \mathrm{e}^{-D_n} a \cdot b] \cdot c^2 = \mathrm{e}^{-\frac{1}{2}D_n}[(D_t a \cdot c) \cdot (\mathrm{e}^{-D_n} c \cdot b) + ac \cdot (D_t \mathrm{e}^{-D_n} c \cdot b)]$$
$$- \mathrm{e}^{-\frac{1}{2}D_n}[(D_t \mathrm{e}^{-D_n} a \cdot c) \cdot cb + (\mathrm{e}^{-D_n} a \cdot c) \cdot (D_t c \cdot b)] \tag{A6}$$

$$(D_t a \cdot b)c - (D_t a \cdot c)b = -a D_t b \cdot c \tag{A7}$$

$$[D_z \mathrm{e}^{-\frac{1}{2}D_n} a \cdot b][\mathrm{e}^{-\frac{1}{2}D_n} c \cdot d] - [\mathrm{e}^{-\frac{1}{2}D_n} a \cdot b][D_z \mathrm{e}^{-\frac{1}{2}D_n} c \cdot d]$$
$$= D_z[\mathrm{e}^{-\frac{1}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{-\frac{1}{2}D_n} c \cdot b] \tag{A8}$$

$$[\mathrm{e}^{\delta D_n} a \cdot b][\mathrm{e}^{-\delta D_n} c \cdot d] = \mathrm{e}^{\delta D_n} ad \cdot cb. \tag{A9}$$

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